Surface Casimir densities on a spherical brane in Rindler-like spacetimes

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Abstract

The vacuum expectation value of the surface energy-momentum tensor is evaluated for a scalar field obeying Robin boundary condition on a spherical brane in $(D+1)$-dimensional spacetime $Ri \times S^{D-1}$, where Ri is a two-dimensional Rindler spacetime. The generalized zeta function technique is used in combination with the contour integral representation. The surface energies on separate sides of the brane contain pole and finite contributions. Analytic expressions for both these contributions are derived. For an infinitely thin brane in odd spatial dimensions, the pole parts cancel and the total surface energy, evaluated as the sum of the energies on separate sides, is finite. For a minimally coupled scalar field the surface energy-momentum tensor corresponds to the source of the cosmological constant type.

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1 Introduction

Motivated by string/M theory, the AdS/CFT correspondence, and the hierarchy problem of particle physics, braneworld models were studied actively in recent years [1]. In this models, our universe is realized as a boundary of a higher dimensional spacetime. In particular, a well studied example is when the bulk is an AdS space. The problem of studying quantum effects in braneworld scenarios is of considerable phenomenological interest, both in particle physics and in cosmology. The braneworld corresponds to a manifold with dynamical boundaries and all fields which propagate in the bulk will give Casimir-type contributions to the vacuum energy, and as a result to the vacuum forces acting on the branes. In dependence of the type of a field and boundary conditions imposed, these forces can either stabilize or destabilize the braneworld. In addition, the Casimir energy gives a contribution to both the brane and bulk cosmological constants and, hence, has to be taken into account in the self-consistent formulation of the braneworld dynamics. Motivated by these, the role of quantum effects on background of Randall–Sundrum geometry has received a great deal of attention. The models with dS and AdS branes, and higher dimensional brane models are considered as well (see, for instance, references given in [2]).

In view of the recent developments in braneworld scenarios, it seems interesting to generalize the study of quantum effects to other types of bulk spacetimes. In particular, it is of interest to consider non-Poincaré invariant braneworlds, both to better understand the mechanism of localized gravity and for possible cosmological applications. Bulk geometries generated by higher-dimensional black holes are of special interest. In these models , the tension and the position of the brane are tuned in terms of black hole mass and cosmological constant and brane gravity trapping occurs in just the same way as in the Randall-Sundrum model. Braneworlds in the background of the AdS black hole were studied in [3]. Like pure AdS space the AdS black hole may be superstring vacuum. It is of interest to note that the phase transitions which can be interpreted as confinement-deconfinement transition in AdS/CFT setup may occur between pure AdS and AdS black hole [4]. Though, in the generic black hole background the investigation of brane-induced quantum effects is technically complicated, the exact analytical results can be obtained in the near horizon and large mass limit when the brane is close to the black hole horizon. In this limit the black hole geometry may be approximated by the Rindler-like manifold (for some investigations of quantum effects on background of Rindler-like spacetimes see [5] and references therein). In the paper [6] we have investigated the Wightman function, the vacuum expectation values of the field square and the energy-momentum tensor for a scalar field with an arbitrary curvature coupling parameter for the spherical brane on the bulk $Ri \times S^{D-1}$, where Ri is a two-dimensional Rindler spacetime. This problem is also of separate interest as an example with gravitational and boundary-induced polarizations of the vacuum, where all calculations can be performed in a closed form. Note that the corresponding quantities induced by a single and two parallel flat branes in the bulk geometry $Ri \times R^{D-1}$ for both scalar and electromagnetic fields are investigated in [7]. For scalar fields with general curvature coupling, in Ref. [8] it has been shown that in the discussion of the relation between the mode sum energy, evaluated as the sum of the zero-point energies for each normal mode of frequency, and the volume integral of the renormalized energy density for the Robin parallel plates geometry it is necessary to include in the energy a surface term concentrated on the boundary (see also the discussion in Ref. [9]). An expression for the surface energy-momentum tensor for a scalar field with a general curvature coupling parameter in the general case of bulk and boundary geometries is derived in Ref. [10]. The vacuum expectation values of the surface energy-momentum tensor on the branes in AdS bulk are investigated in [11]. In particular, it has been shown that the surface densities induced

by quantum fluctuations of bulk fields can serve as a natural mechanism for the generation of cosmological constant in braneworld models of the Randall-Sundrum type with the value in good agreement with recent cosmological observations. The purpose of the present paper is to study the vacuum expectation value of the surface energy-momentum tensor for a scalar field obeying Robin boundary condition on a spherical brane on the bulk $Ri \times S^{D-1}$. The paper is organized as follows. In section 2 we consider the surface energy-momentum tensor and the eigenfunctions for the problem. The vacuum expectation value of the surface energy-momentum tensor in the R-region (the definitions of the R- and L-regions see below) are investigated in section 3. The corresponding quantities for the L-region are discussed in section 4. Section 5 summarizes the main results of the paper.

2 Surface energy-momentum tensor

Consider a real scalar field $\varphi(x)$ on background of $(D+1)$-dimensional spacetime $Ri \times S^{D-1}$, where Ri is a two-dimensional Rindler spacetime. The corresponding line element has the form

$$ds^2 = \xi^2 d\tau^2 - d\xi^2 - r_H^2 d\Sigma_{D-1}^2, \tag{1}$$

with the Rindler-like (τ, ξ) part and $d\Sigma_{D-1}^2$ is the line element for the space with positive constant curvature with the Ricci scalar $R = (D-2)(D-1)/r_H^2$. Line element (1) describes the near horizon geometry of $(D+1)$-dimensional topological black hole with coordinate ξ determining the distance from the horizon. For example, in the case of a $(D+1)$-dimensional Schwarzschild black hole one has $r - r_H = (D-2)\xi^2/4r_H$, where r is the Schwarzschild radial coordinate and $r = r_H$ corresponds to the horizon. For the scalar field $\varphi(x)$ with curvature coupling parameter ζ the dynamics is governed by the field equation

$$\left(\nabla_l \nabla^l + m^2 + \zeta R\right)\varphi = 0, \tag{2}$$

where ∇_l is the covariant derivative operator associated with the corresponding metric tensor g_{ik}. In the cases of minimally and conformally coupled scalars one has $\zeta = 0$ and $\zeta = (D-1)/4D$, respectively. Our main interest in this paper will be the surface Casimir energy and stresses induced on a spherical brane located at $\xi = a$. We will assume that the field satisfies the Robin boundary condition

$$(A_s + n^l \nabla_l)\varphi(x) = 0 \tag{3}$$

on the brane, where A_s is a constant, n^l is the unit inward normal to the brane. This type of conditions is an extension of Dirichlet and Neumann boundary conditions and appears in a variety of situations, including the considerations of vacuum effects for a confined charged scalar field in external fields [12], spinor and gauge field theories, quantum gravity and supergravity [13]. Robin boundary conditions naturally arise for scalar and fermion bulk fields in the Randall-Sundrum model [14]. For boundary condition (3) the vacuum expectation value of the bulk energy-momentum tensor induced by a spherical brane is evaluated in Ref. [6]. In Ref. [10] it was argued that the energy-momentum tensor for a scalar field on manifolds with boundaries in addition to the bulk part contains a contribution located on the boundary. The surface part of the energy-momentum tensor is given by the formula [10]

$$T_{ik}^{(\mathrm{surf})} = \delta(x; \partial M_s)\tau_{ik} \tag{4}$$

where the "one-sided" delta-function $\delta(x; \partial M_s)$ locates this tensor on boundary ∂M_s and

$$\tau_{ik} = \zeta\varphi^2 K_{ik} - (2\zeta - 1/2)h_{ik}\varphi n^l \nabla_l \varphi. \tag{5}$$

Here K_{ik} is the extrinsic curvature tensor for the boundary and h_{ik} is the corresponding induced metric.

Let $\{\varphi_\alpha(x), \varphi_\alpha^*(x)\}$ be a complete set of positive and negative frequency solutions to the field equation (2), obeying boundary condition (3). Here α denotes a set of quantum numbers specifying the solution. By expanding the field operator over the eigenfunctions $\varphi_\alpha(x)$, using the standard commutation rules, for the vacuum expectation value of the surface energy-momentum tensor one finds

$$\langle 0|T_{ik}^{(\mathrm{surf})}|0\rangle = \delta(x;\partial M_s)\langle 0|\tau_{ik}|0\rangle, \quad \langle 0|\tau_{ik}|0\rangle = \sum_\alpha \tau_{ik}\{\varphi_\alpha(x), \varphi_\alpha^*(x)\}, \tag{6}$$

where $|0\rangle$ is the amplitude for the vacuum state, and the bilinear form $\tau_{ik}\{\varphi,\psi\}$ on the right of the second formula is determined by the classical energy-momentum tensor (5). To evaluate the vacuum expectation value of the surface energy-momentum tensor we need the eigenfunctions $\varphi_\alpha(x)$. In the consideration below we will use the hyperspherical angular coordinates $(\vartheta,\phi) = (\theta_1,\theta_2,\ldots,\theta_n,\phi)$ on S^{D-1} with $n = D-2$, $0\leq\theta_k\leq\pi$, $k=1,\ldots,n$, and $0\leq\phi\leq 2\pi$. In these coordinates the variables are separated and the eigenfunctions can be written in the form

$$\varphi_\alpha(x) = C_\alpha f(\xi) Y(m_k;\vartheta,\phi)e^{-i\omega\tau}, \tag{7}$$

where $m_k = (m_0 \equiv l, m_1, \ldots m_n)$, and $m_1, m_2, \ldots m_n$ are integers such that

$$0 \leq m_{n-1} \leq \cdots \leq m_1 \leq l, \quad -m_{n-1} \leq m_n \leq m_{n-1}, \tag{8}$$

$Y(m_k;\vartheta,\phi)$ is the surface harmonic of degree l [15]. The equation for $f(\xi)$ is obtained from field equation (2). The corresponding linearly independent solutions are the Bessel modified functions $I_{\pm i\omega}(\lambda_l\xi)$ and $K_{i\omega}(\lambda_l\xi)$ with the imaginary order, where

$$\lambda_l = \frac{1}{r_H}\sqrt{l(l+n)+\zeta n(n+1)+m^2r_H^2}. \tag{9}$$

The eigenfrequencies are determined from the boundary condition imposed on the field at $\xi = a$. The brane divides the spacetime into two regions with $\xi > a$ (R-region) and $0 < \xi < a$ (L-region). The vacuum properties in these regions are different and we consider them separately.

3 Surface energy in the R-region

For the R-region the unit normal to the boundary and nonzero components of the extrinsic curvature tensor have the form

$$n^l = \delta_1^l, \quad K_{00} = a, \tag{10}$$

and $f(\xi) = K_{i\omega}(\lambda_l\xi)$. For a given $\lambda_l a$, the corresponding eigenfrequencies $\omega = \omega_j = \omega_j(\lambda_l a)$, $j = 1, 2, \ldots$, are determined from boundary condition (3) and are solutions to the equation

$$AK_{i\omega}(x) + xK'_{i\omega}(x) = 0, \quad x = \lambda_l a, \quad A = A_s a, \tag{11}$$

where the prime denotes the differentiation with respect to the argument of the function. For $A_s > 0$ this equation has purely imaginary solutions with respect to ω. To avoid the vacuum instability, below we will assume that $A_s \leq 0$. Under this condition all solutions to (11) are real. The coefficient C_α in Eq. (7) is determined by the normalization condition. Using the relation

$$\int |Y(m_k;\vartheta,\phi)|^2 d\Omega = N(m_k) \tag{12}$$

for the spherical harmonics (the explicit form for $N(m_k)$ will not be necessary in the following consideration), one finds

$$C_\alpha^2 = \frac{1}{r_H^{n+1} N(m_k)} \frac{\bar{I}_{i\omega_j}(\lambda_l a)}{\frac{\partial}{\partial\omega}\bar{K}_{i\omega}(\lambda_l a)|_{\omega=\omega_j}}, \tag{13}$$

where for a given function $F(x)$ we use the notation

$$\bar{F}(x) = AF(x) + xF'(x). \tag{14}$$

Substituting the eigenfunctions into the mode-sum formula (6) and using the relations $K_{i\omega_j}(\lambda_l a)\bar{I}_{i\omega_j}(\lambda_l a) = 1$ and

$$\sum_{m_k} \frac{|Y(m_k;\vartheta,\phi)|^2}{N(m_k)} = \frac{D_l}{S_D}, \tag{15}$$

the vacuum expectation value of the surface energy-momentum tensor can be presented in the form

$$\langle 0|\tau_l^k|0\rangle = \frac{I_{\mathrm{R}}(A)}{2r_H^{D-1} a S_D}\left[2\zeta\delta_l^0\delta_0^k + (4\zeta-1)A\delta_l^k\right], \quad l,k = 0,2,\ldots,D, \tag{16}$$

and $\langle 0|\tau_1^1|0\rangle = 0$, with $S_D = 2\pi^{D/2}/\Gamma(D/2)$ being the total area of the surface of the unit sphere in D-dimensional space, and

$$I_{\mathrm{R}}(A) = \sum_{l=0}^{\infty} D_l \sum_{j=1}^{\infty} \frac{K_{i\omega_j}(\lambda_l a)}{\frac{\partial}{\partial\omega}\bar{K}_{i\omega}(\lambda_l a)|_{\omega=\omega_j}}. \tag{17}$$

Here and below the quantities for the R- and L-regions are denoted by the indices R and L, respectively, and we use the notation

$$D_l = (2l + D - 2)\frac{\Gamma(l + D - 2)}{\Gamma(D-1)l!} \tag{18}$$

for the degeneracy factor. The vacuum expectation value of the surface energy-momentum tensor (16) has a diagonal structure:

$$\langle 0|\tau_l^k|0\rangle = \mathrm{diag}\left(\varepsilon^{(\mathrm{R})}, 0, -p^{(\mathrm{R})}, \ldots, -p^{(\mathrm{R})}\right), \tag{19}$$

with the surface energy density $\varepsilon^{(\mathrm{R})}$, the stress

$$p^{(\mathrm{R})} = \frac{AI_R(A)}{2r_H^{D-1} a}(1-4\zeta), \tag{20}$$

and with the equation of state

$$\varepsilon^{(\mathrm{R})} = -\left[1 + \frac{2\zeta}{A(4\zeta-1)}\right] p^{(\mathrm{R})}. \tag{21}$$

For a minimally coupled scalar field, the latter corresponds to a cosmological constant induced on the brane. Note that the vacuum expectation values of the field square on the brane is also expressed in terms of the function $I_R(A)$:

$$\langle 0|\varphi^2|0\rangle_{\xi=a} = \frac{I_R(A)}{r_H^{D-1} S_D}. \tag{22}$$

The quantity (17) and, hence, the surface energy-momentum tensor diverges and needs some regularization. Many regularization techniques are available nowadays and, depending on the specific physical problem under consideration, one of them may be more suitable than the others. Here we will use the method which is an analog of the generalized zeta function approach. We define the function

$$F_{\rm R}(s) = \sum_{l=0}^{\infty} D_l \zeta_{\rm R}(s, \lambda_l a), \tag{23}$$

where

$$\zeta_{\rm R}(s, \lambda_l a) = \sum_{j=1}^{\infty} \frac{\omega_j^{-s} K_{i\omega_j}(\lambda_l a)}{\frac{\partial}{\partial \omega} \bar{K}_{i\omega}(\lambda_l a)|_{\omega=\omega_j}}. \tag{24}$$

Note that for Dirichlet boundary condition this function vanishes. The computation of vacuum expectation value for the surface energy-momentum tensor requires an analytical continuation of the function $F_{\rm R}(s)$ to the value $s = 0$,

$$I_{\rm R}(A) = F_{\rm R}(s)|_{s=0}. \tag{25}$$

The starting point of our consideration is the representation of the function (24) in terms of contour integral

$$\zeta_{\rm R}(s, x) = \frac{1}{2\pi i} \int_C dz\, z^{-s} \frac{K_{iz}(x)}{\bar{K}_{iz}(x)}, \tag{26}$$

where C is a closed counterclockwise contour in the complex z plane enclosing all zeros $\omega_j(x)$. The location of these zeros enables one to deform the contour C into a segment of the imaginary axis $(-iR, iR)$ and a semicircle of radius R in the right half-plane. We will also assume that the origin is avoided by the semicircle C_ρ with small radius ρ. For sufficiently large s the integral over the large semicircle in (26) tends to zero in the limit $R \to \infty$, and the expression on the right can be transformed to

$$\zeta_{\rm R}(s, x) = \frac{1}{2\pi i} \int_{C_\rho} dz\, z^{-s} \frac{K_{iz}(x)}{\bar{K}_{iz}(x)} - \frac{1}{\pi} \cos \frac{\pi s}{2} \int_{\rho}^{\infty} dz\, z^{-s} \frac{K_z(x)}{\bar{K}_z(x)}. \tag{27}$$

Below we will consider the limit $\rho \to 0$. In this limit the first integral vanishes in the case $s = 0$, and in the following we will concentrate on the contribution of the second integral. For the analytic continuation of this integral we employ the uniform asymptotic expansion of the MacDonald function and its derivative for large values of the order [16]. We will rewrite this expansion in the form

$$K_z(x) \sim \sqrt{\frac{\pi}{2}} \frac{e^{-z\eta(x/z)}}{(x^2 + z^2)^{1/4}} \sum_{q=0}^{\infty} \frac{(-1)^q \tilde{u}_q(t)}{(x^2 + z^2)^{q/2}}, \tag{28}$$

where

$$t = \frac{z}{\sqrt{x^2 + z^2}}, \quad \eta(x) = \sqrt{1 + x^2} + \ln \frac{x}{1 + \sqrt{1 + x^2}}, \quad \tilde{u}_q(t) = \frac{u_q(t)}{t^q}, \tag{29}$$

and the expressions for the functions $u_q(t)$ are given in [16]. From these expressions it follows that the coefficients $\tilde{u}_q(t)$ have the structure

$$\tilde{u}_q(t) = \sum_{m=0}^{q} u_{qm} t^{2m}, \tag{30}$$

with numerical coefficients u_{qm}. From Eq. (28) and the corresponding expansion for the derivative of the MacDonald function we obtain the asymptotic expansion

$$\bar{K}_z(x) \sim -\sqrt{\frac{\pi}{2}}(x^2+z^2)^{1/4}e^{-z\eta(x/z)}\sum_{q=0}^{\infty}\frac{(-1)^q\tilde{v}_q(t)}{(x^2+z^2)^{q/2}}\,, \tag{31}$$

where

$$\tilde{v}_q(t) = \frac{v_q(t)}{t^q} + A\tilde{u}_{q-1}\,, \tag{32}$$

and the expressions for $v_q(t) = t^q\sum_{m=0}^{q} v_{qm}t^{2m}$ are presented in [16]. Note that the functions (32) have the structure

$$\tilde{v}_q(t) = \sum_{m=0}^{q}\tilde{v}_{qm}t^{2m}, \quad \tilde{v}_{qm} = v_{qm} + Au_{q-1,m}\,. \tag{33}$$

From Eqs. (28) and (31) we can find the asymptotic expansion for the ratio in the second integral on the right of formula (27):

$$\frac{K_z(x)}{\bar{K}_z(x)} \sim -\frac{1}{(x^2+z^2)^{1/2}}\sum_{q=0}^{\infty}\frac{(-1)^qU_q(t)}{(x^2+z^2)^{q/2}}\,, \tag{34}$$

where the coefficients $U_q(t)$ are defined by the relation

$$\sum_{q=0}^{\infty}(-1)^q\frac{\tilde{u}_q(t)}{r^q}\left[\sum_{q=0}^{\infty}(-1)^q\frac{\tilde{v}_q(t)}{r^q}\right]^{-1} = \sum_{q=0}^{\infty}\frac{(-1)^qU_q(t)}{r^q}\,, \tag{35}$$

and similar to (30), (33), are polynomials in t:

$$U_q(t) = \sum_{j=0}^{q}U_{qj}t^{2j}. \tag{36}$$

The first three coefficients are given by expressions

$$\begin{aligned} U_0(t) &= 1,\ U_1(t) = \frac{1}{2} - A - \frac{t^2}{2}, \\ U_2(t) &= \frac{3}{8} - A + A^2 - \left(A - \frac{7}{32}\right)t^2 + \frac{49}{576}\,t^4. \end{aligned}$$

Now let us consider the function

$$F_{\mathrm{R}}(s) = -\frac{1}{\pi}\cos\frac{\pi s}{2}\sum_{l=0}^{\infty}D_l\int_{\rho}^{\infty}dz\,z^{-s}\frac{K_z(\lambda_l a)}{\bar{K}_z(\lambda_l a)}. \tag{37}$$

We subtract and add to the integrand in this equation the first N terms of the corresponding asymptotic expansion. This allows us to split (37) into the following pieces

$$F_{\mathrm{R}}(s) = F_{\mathrm{R}}^{(as)}(s) + F_{\mathrm{R}}^{(1)}(s)\,, \tag{38}$$

where

$$F_{\mathrm{R}}^{(as)}(s) = \frac{1}{\pi}\cos\frac{\pi s}{2}\sum_{l=0}^{\infty}D_l\int_{\rho}^{\infty}dz\,z^{-s}\sum_{q=0}^{N}\frac{(-1)^q U_q(t)}{(z^2+\lambda_l^2a^2)^{(q+1)/2}}, \tag{39}$$

$$F_{\mathrm{R}}^{(1)}(s) = -\frac{1}{\pi}\cos\frac{\pi s}{2}\sum_{l=0}^{\infty}D_l\int_{\rho}^{\infty}dz\,z^{-s}\left[\frac{K_z(\lambda_l a)}{\bar{K}_z(\lambda_l a)}+\sum_{q=0}^{N}\frac{(-1)^q U_q(t)}{(z^2+\lambda_l^2a^2)^{(q+1)/2}}\right], \tag{40}$$

and

$$t = z/\sqrt{z^2+\lambda_l^2a^2}. \tag{41}$$

For $N \geq D-1$ the expression for $F_{\mathrm{R}}^{(1)}(s)$ is finite at $s=0$ and, hence, for our aim it is sufficient to subtract $N=D-1$ asymptotic terms. At $s=0$ the function $F_{\mathrm{R}}^{(1)}(s)$ is finite for $\rho=0$ and we can directly put this value. The integral over z in the expression for $F_{\mathrm{R}}^{(as)}(s)$ is finite in the limit $\rho\to 0$ for $0<\mathrm{Re}\,s<1$. For these values we can put $\rho=0$ in Eq. (39). By making use formulae (36), (41), after the integration over z, the asymptotic part is presented in the form

$$F_{\mathrm{R}}^{(as)}(s) = \frac{1}{2\pi}\cos\frac{\pi s}{2}\sum_{q=0}^{N}(-1)^q\left(\frac{r_H}{a}\right)^{q-s}\sum_{j=0}^{q}U_{qj}B\left(j+\frac{1-s}{2},\frac{q+s}{2}\right)\zeta_{S^{D-1}}\left(\frac{q+s}{2}\right), \tag{42}$$

with the beta function $B(x,y)$. In formula (42)

$$\zeta_{S^{D-1}}(z) == \sum_{l=0}^{\infty}D_l\left[(l+D/2-1)^2+b_D\right]^{-z}, \tag{43}$$

is the zeta function for a scalar field on the spacetime $R\times S^{D-1}$ and

$$b_D = \zeta(D-2)(D-1)-(D-2)^2/4+m^2r_H^2. \tag{44}$$

This function is well investigated in literature (see, for example, [17]) and can be presented as a series of incomplete zeta functions. Here we recall that the function $\zeta_{S^{D-1}}(z)$ is a meromorphic function with simple poles at $z=(D-1)/2-j$, where $j=0,1,2,\ldots$ for D even and $0\leq j\leq (D-3)/2$ for D odd. For D even one has $\zeta_{S^{D-1}}(-j)=0$, $j=1,2,\ldots$. In (42), the pole term in the $q=0$ summand comes from the pole of the beta function, whereas in the terms with $q\neq 0$ the pole terms come from the poles of the function $\zeta_{S^{D-1}}(z)$. Laurent-expanding near $s=0$ we find

$$F_{\mathrm{R}}(s) = \frac{F_{\mathrm{R},-1}^{(as)}}{s}+F_{\mathrm{R},0}^{(as)}+F_{\mathrm{R}}^{(1)}(0)+\mathcal{O}(s). \tag{45}$$

Using this result, for the surface energy density induced on the brane one obtains

$$p^{(\mathrm{R})} = p_p^{(\mathrm{R})}+p_f^{(\mathrm{R})}, \tag{46}$$

where for the pole and finite contributions one has

$$\varepsilon_p^{(\mathrm{R})} = \frac{A(4\zeta-1)+2\zeta}{2sr_H^{D-1}aS_D}F_{\mathrm{R},-1}^{(as)}, \tag{47}$$

$$\varepsilon_f^{(\mathrm{R})} = \frac{A(4\zeta-1)+2\zeta}{2r_H^{D-1}aS_D}\left[F_{\mathrm{R},0}^{(as)}+F_{\mathrm{R}}^{(1)}(0)\right]. \tag{48}$$

The corresponding formulae for the pole and finite parts of the surface stress are obtained by using the equation of state (21). The surface energy can be found integrating the energy density,

$$E^{(\mathrm{R,surf})} = \int d^D x \sqrt{|g|} \langle 0| T_0^{(\mathrm{surf})0} |0\rangle = a r_H^{D-1} S_D \varepsilon^{(\mathrm{R})}. \tag{49}$$

The pole and finite parts of the vacuum expectation value of the field square on the brane are obtained by the formulae (22), (25), (45).

4 Surface densities in the L-region

In this section we consider the region between the horizon and the brane, $0 < \xi < a$ (L-region), for which one has $n^l = -\delta_1^l$ and $K_{00} = -a$. As in the previous section we will assume that the field obeys boundary condition (3) on the surface $\xi = a$. To deal with discrete spectrum, we can introduce the second brane located at $\xi = b < a$, on whose surface we impose boundary conditions as well. After the construction of the corresponding zeta function we take the limit $b \to 0$. As a result, we can see that the surface energy-momentum tensor in the L-region has the structure given by (19) and with the equation of state (21). For the surface energy density one obtains the expression

$$\varepsilon^{(\mathrm{L})} = \frac{A(4\zeta - 1) + 2\zeta}{2 r_H^{D-1} a S_D} I_{\mathrm{L}}(A), \quad A = -aA_s, \tag{50}$$

where now $I_{\mathrm{L}}(A) = F_{\mathrm{L}}(s)|_{s=0}$ with

$$F_{\mathrm{L}}(s) = -\frac{1}{\pi} \cos\frac{\pi s}{2} \sum_{l=0}^{\infty} D_l \int_{\rho}^{\infty} dz\, z^{-s} \frac{I_z(\lambda_l a)}{\bar{I}_z(\lambda_l a)}. \tag{51}$$

For a given A this expression differs from the corresponding expression for the R-region by the replacement $K_z(x) \to I_z(x)$. Note that the similar relation takes place for the bulk energy-momentum tensor as well. As in the previous section, to avoid the vacuum instability, here we assume that $A_s \leq 0$. Under this condition, for a given $\lambda_l a$ the function $\bar{I}_z(\lambda_l a)$ has no real positive zeros with respect to z. The uniform asymptotic expansion for the integrand in (51) is obtained from the corresponding formula with the functions $K_z(\lambda_l a)$ (see formula (34)) by the replacement

$$(-1)^q U_q(t) \to -U_q(t). \tag{52}$$

The vacuum stress is a sum of pole and finite parts

$$\varepsilon^{(\mathrm{L})} = \varepsilon_p^{(\mathrm{L})} + \varepsilon_f^{(\mathrm{L})}, \tag{53}$$

with

$$\begin{aligned} \varepsilon_p^{(\mathrm{L})} &= \frac{A(4\zeta - 1) + 2\zeta}{2 s r_H^{D-1} a S_D} F_{\mathrm{L},-1}^{(as)}, \\ \varepsilon_f^{(\mathrm{L})} &= \frac{A(4\zeta - 1) + 2\zeta}{2 r_H^{D-1} a S_D} \left[F_{\mathrm{L},0}^{(as)} + F_{\mathrm{L}}^{(1)}(0) \right]. \end{aligned} \tag{54}$$

The formulae for $F_{\mathrm{L},-1}^{(as)}$, $F_{\mathrm{L},0}^{(as)}$, $F_{\mathrm{L}}^{(1)}(0)$ are obtained from the corresponding expressions for the R-region by the replacements $K_z(x) \to I_z(x)$ and (52). In particular,

$$F_{\mathrm{L}}^{(as)}(s) = -\frac{1}{2\pi} \cos\frac{\pi s}{2} \sum_{q=0}^{N} \left(\frac{r_H}{a}\right)^{q-s} \sum_{j=0}^{q} U_{qj} B\left(j + \frac{1-s}{2}, \frac{q+s}{2}\right) \zeta_{S^{D-1}}\left(\frac{q+s}{2}\right). \tag{55}$$

The surface energy density is related to the stress by formula (21) with the replacement R→L and for the total surface energy one has

$$E^{(\mathrm{L,surf})} = ar_H^{D-1} S_D \varepsilon^{(\mathrm{L})}. \tag{56}$$

The vacuum expectation value of the field square on the brane for the L-region is also expressed in terms of the function $I_{\mathbf{L}}(A)$. For an infinitely thin brane taking the R- and L-regions together, the pole parts of the surface energy densities cancel for odd values of the spatial dimension D. In this case the total surface energy $E^{(\mathrm{surf})} = E^{(\mathrm{R,surf})} + E^{(\mathrm{L,surf})}$ is finite and can be directly evaluated by the formula

$$\begin{aligned} E^{(\mathrm{surf})} &= \frac{A(1-4\zeta)+2\zeta}{2\pi} \left\{ \sum_{k=0}^{N_1} \left(\frac{r_H}{a}\right)^{2k+1} \zeta_{S^{D-1}} \left(k+\frac{1}{2}\right) \sum_{j=0}^{2k+1} U_{2k+1,j} B\left(j+\frac{1}{2}, k+\frac{1}{2}\right) \right. \\ &\left. + \sum_{l=0}^{\infty} D_l \int_0^{\infty} dz \left[\frac{I_z(\lambda_l a)}{\bar{I}_z(\lambda_l a)} + \frac{K_z(\lambda_l a)}{\bar{K}_z(\lambda_l a)} - \sum_{k=0}^{N_1} \frac{2U_{2k+1}(t)}{(z^2+\lambda_l^2 a^2)^{k+1}} \right] \right\}, \end{aligned} \tag{57}$$

where $N_1 = [(N-1)/2]$, $N \geq D-1$, and t is defined by relation (41). Note that the cancellation of the pole terms coming from oppositely oriented faces of infinitely thin smooth boundaries takes place in vary many situations encountered in the literature. It is a simple consequence of the fact that the second fundamental forms are equal and opposite on two faces of each boundary and, consequently, the values of the corresponding coefficient in the heat kernel expansion summed over two faces of each boundary vanishes.

We have investigated the surface densities for both R- and L-regions. In the corresponding braneworld scenario the geometry is made up by two slices of the region $0 < \xi < a$ glued together at the brane with a orbifold-type symmetry condition analogous to that in the Randall-Sundrum model (see, for instance, [6]). For an untwisted scalar field the coefficient A_s in the boundary condition is related to the brane mass parameter c of the field and the extrinsic curvature of the brane by the relation $A_s = (c - \zeta/a)/2$. For a twisted scalar Dirichlet boundary condition is obtained. It should be noted that in the orbifolded version due to Z_2 symmetry the extrinsic curvature tensor is the same on both sides of the fixed point and the cancellation of the pole terms for odd values D does not take place. A natural way to deal with surface divergences is to consider more realistic brane models with finite thickness. As it has been discussed in [18] for de Sitter brane model, the finite thickness of the brane regularizes the ultraviolet behavior and acts as a natural cutoff.

5 Conclusion

In this paper we have investigated the surface Casimir densities induced on a spherical brane in the Rindler-like spacetime $Ri \times S^{D-1}$ by quantum fluctuations of a scalar field with an arbitrary curvature coupling parameter. The corresponding volume vacuum expectation values of the energy–momentum tensor were investigated in [6]. We consider a scalar field with Robin boundary conditions and as a regularization method the zeta function technique is employed. The spherical brane divides the background space into two regions, referred as R- and L-regions. We have constructed an integral representations for the corresponding zeta functions in both these regions, which are well suited for the analytic continuation. Subtracting and adding to the integrands the leading terms of the corresponding uniform asymptotic expansions, we present the corresponding functions as a sum of two parts. The first one is convergent at the physical point

and can be evaluated numerically. In the second, asymptotic part the pole contributions are given explicitly in terms of the zeta function for a scalar field on the spacetime $R\times S^{D-1}$. The latter is well-investigated in literature. As a consequence, the vacuum expectation values of the surface energy-momentum tensor for separate R- and L-regions contain pole and finite contributions. The remained pole term is a characteristic feature for the zeta function regularization method and has been found for many other cases of boundary geometries. For a minimally coupled scalar field, the surface energy-momentum tensor induced by quantum vacuum effects corresponds to a source of a cosmological constant type located on the brane. In odd spatial dimensions in the case of an infinitely thin brane, taking the R- and L-regions together, the pole parts of the surface vacuum energies cancel. As a result the total surface energy is finite and is determined by formula (57) with the function $U_q(t)$ is defined by relation (35). The results obtained here can be applied to the braneworld in the AdS black hole bulk in the limit when the brane is close to the black hole horizon. In this paper we have considered the surface energy-momentum tensor on a codimension one smooth brane. For non-smooth boundaries an additional part in the energy-momentum tensor arises located on corners. The corresponding corner terms can be important in codimension two braneworld scenarios (see, for instance, [19] and references therein).

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References

[1] P. Horava and E. Witten, Nucl. Phys. **B460**, 460 (1996); L. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. **B436**, 257 (1998); N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. **B436**, 263 (1998); L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).

[2] A.A. Saharian, Nucl. Phys. **B712**, 196 (2005).

[3] P. Krauss, JHEP **9912**, 011 (1999); D. Ida, JHEP **0009**, 014 (2000); C. Barcelo and M. Visser, Phys. Lett. **B482**, 183 (2000); H. Stoica, S.-H. Henry Ty, and I. Wasserman, Phys. Lett. **B482**, 205 (2000); C. Gomez, B. Janssen, and P.J. Silva, JHEP **0004**, 027 (2000); A. Kamenshchik, U. Moshella, and V. Pasquier, Phys. Lett. **B487**, 7 (2000); P. Bowcock, C. Charmousis, and R. Gregory, Class. Quantum Grav. **17**, 4745 (2000); D. Birmingham and M. Rinaldi, Mod. Phys. Lett. **A16**, 1887 (2001); C. Csaki, J. Erlich, and C. Grojean, Nucl. Phys. **B604**, 312 (2001); S. Nojiri, S. D. Odintsov, and S. Ogushi, Int. J. Mod. Phys **A17**, 4809 (2002); S. Nojiri and S. D. Odintsov, Int. J. Mod. Phys **A18**, 2001 (2002); S. Nojiri and S. D. Odintsov, Phys. Lett. **B562**, 9 (2003).

[4] E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998).

[5] A.A. Bytsenko, G. Cognola, and S. Zerbini, Nucl. Phys. **B458**, 267 (1996); S. Zerbini, G. Cognola, and L. Vanzo, Phys. Rev. D **54**, 2699 (1996); G. Cognola, E. Elizalde, and S. Zerbini, Phys. Lett. B **585**, 155 (2004).

[6] A. A. Saharian and M. R. Setare, Nucl. Phys. B, in press, hep-th/0505224.

[7] P. Candelas and D. Deutsch, Proc. R. Soc. London **354**, 79 (1977); A. A. Saharian, Class. Quantum Grav. **19**, 5039 (2002); R. M. Avagyan, A. A. Saharian, and A. H. Yeranyan, Phys. Rev. D **66**, 085023 (2002); A. A. Saharian, R. S. Davtyan, and A. H. Yeranyan, Phys. Rev. D **69**, 085002 (2004); A. A. Saharian and M. R. Setare, Class. Quantum Grav. **21**, 5261 (2004); A. A. Saharian, R. M. Avagyan, and R. S. Davtyan, Int. J. Mod. Phys. Lett. A, in press, hep-th/0504189.

[8] A. Romeo and A. A. Saharian, J. Phys. A **35**, 1297 (2002).

[9] S. A. Fulling, J. Phys. A **36**, 6857 (2003).

[10] A. A. Saharian, Phys. Rev. D **69**, 085005 (2004).

[11] A. A. Saharian, Phys. Rev. D **70**, 064026 (2004).

[12] J. Ambjørn and S. Wolfram, Ann. Phys. (N.Y.) **147**, 33 (1983).

[13] G. Esposito, A. Yu. Kamenshchik, and G. Polifrone, *Euclidean Quantum Gravity on Manifolds with Boundary* (Kluwer, Dordrecht, 1997).

[14] T. Gherghetta and A. Pomarol, Nucl. Phys. **B586**, 141 (2000).

[15] A. Erdélyi et al., *Higher Transcendental functions. Vol. 2* (McGraw Hill, New York, 1953).

[16] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical functions* (National Bureau of Standards, Washington D.C., 1964).

[17] R. Camporesi, Phys. Rep. **196**, 1 (1990).

[18] M. Minamitsuji, W. Naylor, M. Sasaki, Nucl. Phys. **B737**, 121 (2006); M. Minamitsuji, W. Naylor, M. Sasaki, Phys. Lett. **B633**, 607 (2006).

[19] Y. Aghababaie et al., JHEP **0309**, 037 (2003).